§ Space of linear transformations.

Prop.: Let V and W be vector spaces over F.

Then the set $\mathcal{L}(V, W)$ of all linear transformations from V to W is a vector space over F under the following operations:

- For linear $T, U: V \to W$, we define $T+U: V \to W$ by $(T+U)(\vec{x}) := T(\vec{x}) + U(\vec{x})$
- For any $a \in F$, we define $aT: V \to W$, $aT(\vec{x}) = a \cdot T(\vec{x})$

Pf: Exercise.

Lemma: Let V and W be finite-dim vector spaces with ordered bases β and γ, resp.
Let $T, U : V \to W$ linear transformation

Then
- $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$
- $[aT]_\beta^\gamma = a \cdot [T]_\beta^\gamma \quad \forall a \in F.$

pf: $$[T+U]_\beta^\gamma = \begin{pmatrix} & | & \\ \cdots & [(T+U)(\vec{v}_j)]_\gamma & \cdots \\ & | & \end{pmatrix} = \begin{pmatrix} & | & \\ \cdots & [T(\vec{v}_j)]_\gamma & \cdots \\ & | & \end{pmatrix} + \begin{pmatrix} & | & \\ \cdots & [U(\vec{v}_j)]_\gamma & \cdots \\ & | & \end{pmatrix}$$

$$= [T]_\beta^\gamma + [U]_\beta^\gamma$$

□

Theorem: Let V and W be finite-dim vector spaces over F with dimensions n and m, and ordered bases β and γ, resp.

Then, the map $\Phi: \mathcal{L}(V,W) \xrightarrow{\cong} M_{m\times n}(F)$

$$T \rightsquigarrow [T]_\beta^\gamma$$

is an isomorphism.

Cor: $\dim \mathcal{L}(V,W) = \dim M_{m\times n}(F) = \dim V \cdot \dim W$.

Pf of Thm: $\cdot$ Φ is linear: $\Phi(T+U) = [T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$

$$= \Phi(T) + \Phi(U)$$

$$\Phi(a \cdot T) = [aT]_\beta^\gamma = a \cdot [T]_\beta^\gamma$$
$$= a \cdot \Phi(T)$$

$\cdot$ Φ is bijective: For any $A = (a_{ij}) \in M_{m\times n}(F)$, want to show

$\exists$ unique $T: V \to W$ s.t. $\Phi(T) = [T]_\beta^\gamma = A$.

Let $\beta = \{\vec{v}_1, \cdots, \vec{v}_n\}$, $\gamma = \{\vec{w}_1, \cdots, \vec{w}_m\}$ basis for V and W, resp.

$\Rightarrow$ $\exists!$ $T: V \to W$ s.t.

$T(\vec{v}_j) = \sum_{i=1}^{m} a_{ij} \vec{w}_i$ $\forall j = 1, \cdots, n$. Then $\Phi(T) = \left(\cdots [T(v_j)]_\gamma \cdots \right) = A$

$\square$

Hence, given $T: V \longrightarrow W$ basis β, γ resp.

have the following Commutative diagram :

$T \in \mathcal{L}(V,W)$

$$\begin{tikzcd}
V \arrow[rr, "T"] \arrow[d, "\phi_\beta = [\cdot]_\beta"'] & & W \arrow[d, "\phi_\gamma = [\cdot]_\gamma"] \\
F^n \arrow[rr, "L_A"'] & & F^m
\end{tikzcd}$$

$$\begin{tikzcd}
\vec{v} \arrow[r, squiggly] \arrow[d, squiggly] & \vec{w} \arrow[d, squiggly] \\
\phi_\beta(\vec{v}) = [\vec{v}]_\beta \arrow[r, squiggly] & \phi_\gamma(w) = [w]_\gamma
\end{tikzcd}$$

$$T \overset{\Phi}{\longmapsto} L_A$$

$L_A \in M_{m\times n}(F)$

where $A = [T]_\beta^\gamma \in M_{m\times n}(F)$

§ Change of Coordinate

Given ordered basis β and β' for a vector space V

Let $\boxed{Q = [I_V]_{\beta'}^{\beta}}$. Assume $\beta = \{\vec{v_1}, \cdots, \vec{v_n}\}$, $\beta' = \{\vec{v_1'}, \cdots, \vec{v_n'}\}$

then

$$Q = \begin{pmatrix} | & & | \\ [I_V(\vec{v_1'})]_\beta & \cdots & [I_V(\vec{v_n'})]_\beta \\ | & & | \end{pmatrix} = \begin{pmatrix} | & & | \\ [\vec{v_1'}]_\beta & \cdots & [\vec{v_n'}]_\beta \\ | & & | \end{pmatrix}.$$

Def: The matrix $Q = [I_V]_{\beta'}^{\beta}$, is called the change of coordinate matrix.

It changes β'-coord to β-coord.

Prop: (a) Q is invertible.

(b) For all $\vec{v} \in V$. $\boxed{[\vec{v}]_{\beta} = Q \cdot [\vec{v}]_{\beta'}}$

$$
\begin{tikzcd}
V \arrow[r, "I_V"] \arrow[d, "\phi_{\beta'}"'] & V \arrow[l, bend left, "I_V"] \arrow[d, "\phi_{\beta}"] \\
F^n \arrow[r, "{Q := [I_V]_{\beta'}^{\beta}}"] & F^n \arrow[l, bend left, "{[I_V]_{\beta}^{\beta'} = Q^{-1}}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\vec{v} \arrow[r, squiggly] \arrow[d, squiggly] & \vec{v} \arrow[d, squiggly] \\
{[\vec{v}]_{\beta'}} & {[\vec{v}]_{\beta}}
\end{tikzcd}
$$

Pf: (a) Since I_V is invertible, Q is invertible matrix.

(b) Let $\vec{v} \in V$. Then $[\vec{v}]_{\beta} = [I_V(\vec{v})]_{\beta} = [I_V]_{\beta'}^{\beta} \cdot [\vec{v}]_{\beta'}$

∎

Example: $V = \mathbb{R}^3$. $\beta = \left\{ \underset{\vec{v}_1}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix}0\\1\\0\end{pmatrix}}, \underset{\vec{v}_3}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \right\}$

$\beta' = \left\{ \begin{pmatrix}1\\1\\0\end{pmatrix} = \vec{v}_1{}', \begin{pmatrix}1\\-1\\0\end{pmatrix} = \vec{v}_2{}', \begin{pmatrix}0\\0\\1\end{pmatrix} = \vec{v}_3{}' \right\}$

$\vec{v}_1{}' = \vec{v}_1 + \vec{v}_2$

Then $Q = [I_V]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$[\vec{v}]_\beta = Q \cdot [\vec{v}]_{\beta'}$

$\Rightarrow [\vec{v}]_{\beta'} = Q^{-1} \cdot [\vec{v}]_\beta$

Note $\vec{v}_1 = \frac{1}{2}\vec{v}_1{}' + \frac{1}{2}\vec{v}_2{}'$

$Q^{-1} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [I_V]_{\beta}^{\beta'}$

Let $\vec{v} = \begin{pmatrix}2\\3\\4\end{pmatrix} \in \mathbb{R}^3$. $\Leftrightarrow [\vec{v}]_\beta = \begin{pmatrix}2\\3\\4\end{pmatrix}$. Then $[\vec{v}]_{\beta'} = Q^{-1} \cdot [v]_\beta = \begin{pmatrix}5/2\\-1/2\\4\end{pmatrix}$ $\square$

In short summary: Given β, β' ordered bases of V.

$$[v]_\beta = Q[v]_{\beta'} \quad , \quad \text{where } Q = [I_V]_{\beta'}^{\beta}$$

Theorem: Let T be a linear operator on a finite-dim vector space V,

i.e., $T: V \longrightarrow V$, and let β and β' be ordered basis for V.

Suppose Q is the change coord matrix that changes β'-coord to β-coord.

Then, $\boxed{[T]_{\beta'} = Q^{-1}[T]_\beta Q.}$

$= [L_A]_\beta$ ⇜ standard basis $= \{\vec{x}_1, \cdots, \vec{x}_n\}$

Rmk (1). Let $A \in M_{n\times n}(F)$. and γ be an ordered basis for F^n.

Then, $[L_A]_\gamma = \underbrace{Q^{-1} A Q}_{\text{Similar/Conjugate matrix}}$, where $Q = \begin{pmatrix} | & & | \\ \vec{x}_1, & \cdots, & \vec{x}_n \\ | & & | \end{pmatrix} = [I_V]_\gamma^\beta$

(2). $[T]_{\beta'} = Q^{-1}[T]_\beta Q \iff [T]_\beta = Q \cdot [T]_{\beta'} \cdot Q^{-1}$

(3). More generally, $T: V \to W$. β, β' basis for V. γ, γ' basis for W.

Then $[T]_{\beta'}^{\gamma'} = \underbrace{(Q_W)^{-1}}_{= [I_W]_\gamma^{\gamma'}} [T]_\beta^\gamma \cdot \underbrace{Q_V}_{= [I_V]_{\beta'}^\beta}$ $(*)$

Proof of Theorem.

First pf: By Computation

$$Q\cdot[T]_{\beta'} = [I_V]^{\beta}_{\cancel{\beta'}}\cdot[T]^{\cancel{\beta'}}_{\beta'} = [I_V\circ T]^{\beta}_{\beta'}$$

$$= [T\circ I_V]^{\beta}_{\beta'}$$

$$= [T]^{\beta}_{\cancel{\beta}}\cdot[I_V]^{\cancel{\beta}}_{\beta'} = [T]_{\beta}\cdot Q.$$

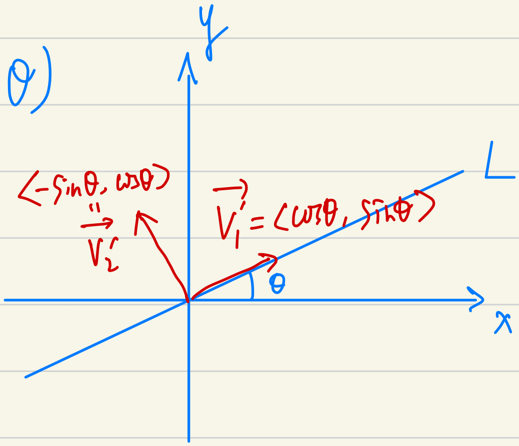
Second pf: Use Commutative diagram

(pf $*$)

$$
\begin{tikzcd}
V \arrow[r, "I_V"] \arrow[d, "\phi_{\beta'}"'] \arrow[rrr, bend left, "T"] & V \arrow[r, "T"] \arrow[d, "\phi_{\beta}"'] & W \arrow[r, "I_W"] \arrow[d, "\phi_{\gamma}"] & W \arrow[d, "\phi_{\gamma'}"] \\
F^n \arrow[r, "{[I_V]_{\beta'}^{\beta}}"] \arrow[rrr, bend right, "{[T]_{\beta'}^{\gamma'}}"'] & F^n \arrow[r, "{[T]_{\beta}^{\gamma}}"] & F^m \arrow[r, "{[I_W]_{\gamma}^{\gamma'}}"] & F^m
\end{tikzcd}
$$

$$\Rightarrow \quad [T]_{\beta'}^{\gamma'} = [I_W]_{\gamma}^{\gamma'} \cdot [T]_{\beta}^{\gamma} \cdot [I_V]_{\beta'}^{\beta}$$

Example 1: $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ projection to line L $(\varphi = \theta)$



Consider the basis $\beta' = \left\{ \overset{\vec{v}_1' =}{\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}}, \overset{\vec{v}_2' =}{\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}} \right\}$

Then $T(\vec{v}_1') = v_1'$ $\quad T(\vec{v}_2') = 0$.

$$\Rightarrow \quad [T]_{\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Change of basis matrix $Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\qquad Q^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Hence $[T]_\beta = Q \cdot [T]_{\beta'} \cdot Q^{-1} = \begin{pmatrix} \cos^2\theta & \cos\theta \cdot \sin\theta \\ \cos\theta \cdot \sin\theta & \sin^2\theta \end{pmatrix}$

$$\begin{array}{cc} \nearrow & \nearrow \\ [T(e_1)]_\beta & [T(e_2)]_\beta \end{array}$$

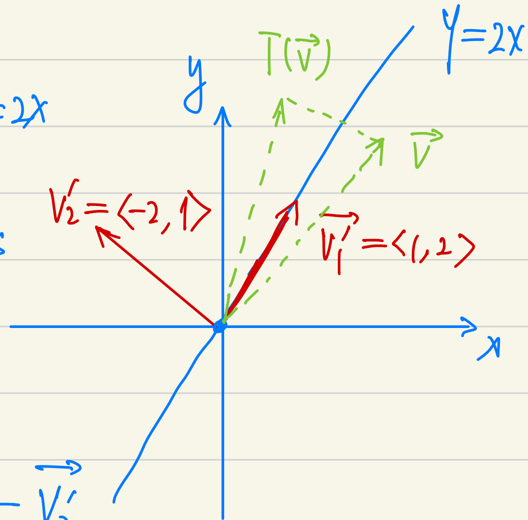
$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

Example 2. $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ reflection about the line $y=2x$

Want to compute $[T]_\beta$ where β is the standard basis

Consider the basis $\beta' = \{ \underset{=\vec{v}_1'}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}, \underset{=\vec{v}_2'}{\begin{pmatrix} -2 \\ 1 \end{pmatrix}} \}$

Then $T(\vec{v}_1') = \vec{v}_1'$ and $T(\vec{v}_2') = -\vec{v}_2'$



$$\Rightarrow [T]_{\beta'} = \left([T(\vec{v}_1')]_{\beta'}, [T(\vec{v}_2')]_{\beta'} \right) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The change of basis matrix Q from β' to β is

$$Q = [I]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \Rightarrow Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

Hence $[T]_{\beta} = Q \cdot [T]_{\beta'} Q^{-1} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}$.

$\square$